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2009 J. Phys. A: Math. Theor. 42 165004

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The matrix product solution of the multispecies partially asymmetric exclusion process

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Received 5 January 2009, in final form 2 March 2009

Published 27 March 2009

Online at stacks.iop.org/JPhysA/42/165004

Abstract

We find the exact solution for the stationary state measure of the partially asymmetric exclusion process on a ring with multiple species of particles. The solution is in the form of a matrix product representation where the matrices for a system of N species are defined recursively in terms of the matrices for a system of $N-1$ species. A complete proof is given, based on the quadratic relations verified by these matrices. This matrix product construction is interpreted in terms of the action of a transfer matrix.

PACS numbers: 05.40.-a, 05.70.Ln, 02.50.-r

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Particles hopping on a one-dimensional lattice with hard-core exclusion interactions provide a simple framework for the study of interacting many-body systems [1, 2]. In particular, when the particle hopping is asymmetric, a macroscopic particle current results and the system attains a nonequilibrium steady state (NESS) in which detailed balance is not satisfied (see [3] for a recent review).

A fundamental example of such a system is the asymmetric simple exclusion process (ASEP) [4, 5]. Here particles attempt hops to the right neighbour site with unit rate and to the left neighbour site with rate q . The hop is carried out when the destination site is empty. The special case, $q = 0$, is known as the totally asymmetric simple exclusion process (TASEP). With periodic boundary conditions the NESS of the ASEP has a very simple form (all allowed configurations of particles are equally likely) yet dynamical properties such as diffusion of a tagged particle [6–9] and large deviations of the current [10] have proved to be non-trivial.

When open boundary conditions, where particles attempt to enter and exit at the left and the right boundary, are used instead of periodic, the NESS takes on a non-trivial form. It may be represented by a matrix product state [11] in which the steady-state probabilities for each configuration are obtained from the products of two matrices D and E according to whether each site is occupied or empty in the configuration. These matrices obey a quadratic algebra which provides a motif from which all the steady-state probabilities may be generated. This quadratic algebra is related to the q -deformed harmonic oscillator [7, 12–16].

A generalization of the ASEP is to the case of two species of particles. A well-studied model is that of usual (first-class) and second-class particles. In this case, both first- and second-class particles hop to the right with unit rate and to the left with rate q . However, if the site to the right of a first-class particle is occupied by a second-class particle the first- and second-class particles exchange places with rate 1. Conversely, if the site to the left of a first-class particle is occupied by a second-class particle they can exchange with rate q . Thus a second-class particle behaves as a hole from the point of view of a first-class particle but behaves as a particle from the point of view of a hole. When q is lower than 1, the second-class particles will move forwards in an environment of a low density of first-class particles but backwards in a high-density environment; therefore, the introduction of a second-class particle is a useful tool in the study of the microscopic structure of shocks [17–21]. The stationary state of a periodic system containing second- and first-class particles has been obtained using the matrix product formulation by Derrida *et al* [20]. Here the three matrices D , E and A (corresponding to first-class particles, second-class particles and holes, respectively) obey a quadratic algebra closely related to that of [11]. Quadratic algebras for a two-species exclusion process have been further explored [22–24], and a general classification of quadratic algebras has been made in [25]. The range of matrix product states and quadratic algebras relevant to the exclusion process and other stochastic systems have been summarized in [3].

A natural generalization of the two-species case of first- and second-class particles is to the multispecies process where there is a hierarchy amongst the different species. That is, for a system with N species, the N th class particles are treated by all other classes of particle as holes, $(N-1)$ th class particles treat N th class particles as holes but are themselves treated as holes by first-class, second-class, . . . , $(N-2)$ th class particles, and so on, up to the first-class particles which treat all other species as holes. We refer to this model as the N -ASEP. In the physics literature, the $N = 3$ totally asymmetric case was considered by Mallick *et al* [26], and a matrix product steady state was determined. It was shown that the matrices obeyed more complicated relations than the quadratic algebra of the two-species case.

In the probabilistic literature, Ferrari and Martin [27] provided a construction for the N -TASEP whereby the dynamics is related to the dynamics of an N -line process i.e. N -coupled TASEPs. The steady state for the N -line process has each of its configurations equally likely. Therefore, to sample the configurations of the N -TASEP according to their steady-state probabilities one picks from a uniform distribution an N -line configuration and projects this onto the corresponding N -TASEP configuration. Ferrari and Martin also showed that the N -line construction could be interpreted as a system of N queues in tandem with priority customers.

In a recent work [28], it was shown that how the construction of Ferrari and Martin can be inverted, and the steady-state probabilities written down as matrix products. A proof was given for the case $N = 3$, and the matrices for general N were written down in a hierarchical fashion.

In the present work, we generalize the solution to the partially asymmetric case and provide the full proof of the matrix product state for arbitrary N . Let us summarize the key points of our solution. The matrices $X_J^{(N)}$, appearing in the matrix product expression

for the steady state (24), depend on J , the species present at the site to which the matrix corresponds and N the number of species represented in the system. The matrices are defined in equation (25) in a hierarchical way: the matrices for an N -species system are expressed in terms of those for an $(N-1)$ -species system. This, in turn, means that the weights of the N -species system may be expressed in terms of the weights of an $(N-1)$ -species system via a transfer matrix which is defined in equation (44). For the two-species case ($N = 2$) the matrices obey a quadratic algebra (10)–(12) and this allows reduction relations which relate the weights of a system of L sites to those of a system of $L - 1$ sites. However, for $N > 3$ the quadratic algebra is replaced by a more complicated set of relations (69)–(71) involving additional ‘hat’ matrices, again defined hierarchically in equations (73), (74). Both the matrices $X_J^{(N)}$ and the hat matrices $\hat{X}_J^{(N)}$ are expressed recursively in terms of auxiliary matrices $a_{JM}^{(N)}$ defined in equations (27)–(33). These auxiliary matrices are themselves tensor products of the four fundamental matrices $\mathbb{1}, \delta, \epsilon, A$ which appear in the $N = 2$ solution. The algebraic properties of the auxiliary matrices, in particular the symmetry relation (57) and the commutation relations (61)–(63), are the key to the proof of the quadratic relations obeyed by the matrices $X_J^{(N)}$ and $\hat{X}_J^{(N)}$ which, in turn, furnish the proof of the matrix product representation of the stationary state.

The transfer matrix mentioned above allows us to investigate how the construction of Ferrari and Martin generalizes to the partially asymmetric case. For the TASEP, the transfer matrix implements the Ferrari–Martin construction explicitly. However, in the system with partial asymmetry, the queuing process interpretation does not hold anymore and is replaced by a more general recurrence between systems with N and $N - 1$ species.

The paper is structured as follows: in section 2, we define the multispecies asymmetric exclusion process. In section 3, we write the matrix product representation of the stationary state of this model. In section 4, we give an interpretation of this matrix product representation in terms of a transfer matrix. Then, after writing the algebraic relations obeyed by the auxiliary matrices related to the matrix product representation in section 5, we give a complete proof of the matrix product expression in section 6. Appendix A is devoted to the calculation of some traces of product of matrices. Appendix B proves a classification of nonzero elements of the transfer matrix, while appendix C contains the proof of a special case of the identities required to ensure that the matrix product expression is valid.

2. Multispecies ASEP

We consider the multispecies asymmetric simple exclusion process with both forward and backward jumps on a one-dimensional lattice with periodic boundary conditions. This stochastic model is defined on a configuration space such that each of the L sites of the lattice can be occupied by at most one particle (exclusion rule). Each particle has a label which is an integer between 1 to N , the ‘class’ of the particle. (We use the terms ‘class’ and ‘species’ interchangeably.) The unoccupied sites (holes) will be considered as particles of class 0. The stochastic dynamics can be expressed in terms of exchanges of particles at neighbouring sites. The transitions which can occur depending on the classes of both particles are

$$JK \rightarrow KJ \quad \text{with rate } 1 \quad \text{if } 1 \leq J < K \leq N \tag{1}$$

$$KJ \rightarrow JK \quad \text{with rate } q \quad \text{if } 1 \leq J < K \leq N \tag{2}$$

$$J0 \rightarrow 0J \quad \text{with rate } 1 \quad \text{if } 1 \leq J \leq N \tag{3}$$

$$0J \rightarrow J0 \quad \text{with rate } q \quad \text{if } 1 \leq J \leq N. \tag{4}$$

All classes of particles jump to the right with rate 1 and to the left with rate q if the destination site is empty. When two particles of different classes are on neighbouring sites, they can exchange with rate 1 if the particle with the smallest class is on the left, and with rate q if it is on the right. Thus, for a particle of class r , all the particles of class larger than r behave as holes. At this point, it might seem natural to consider holes as particles of class $N + 1$ rather than 0. However, we do not adopt this convention as it would make the expression of the stationary state more complicated in the following.

We use the site variable $\tau_l = 0, 1, \dots, N$. If $\tau_l = 0$ the site is empty; if $\tau_l = r > 0$ site l contains a r th class particle. Let us denote by $\vec{\tau} = (\tau_1, \dots, \tau_L)$ a configuration of the system. The dynamics of the system can be encoded in a Markov matrix M . The time evolution of the probability $P_t(\vec{\tau})$ to be in a configuration $\vec{\tau}$ at time t is given by the master equation

$$\frac{d}{dt}P_t(\vec{\tau}) = \sum_{\vec{\tau}'} M(\vec{\tau}, \vec{\tau}')P_t(\vec{\tau}'). \quad (5)$$

The matrix M is a $(N + 1)^L$ by $(N + 1)^L$ matrix which acts on the configuration space. As the numbers of particles of each class are conserved by the dynamics, we will restrict ourselves to a configuration space with fixed number of particles and holes. We call P_r the number of particles of class r . The restricted configuration space $\Omega(P_0, P_1, \dots, P_N)$ has dimension

$$|\Omega| = \frac{L!}{P_0!P_1!\dots P_N!}, \quad (6)$$

and the restricted Markov matrix which acts on it is $|\Omega|$ by $|\Omega|$.

3. Matrix product formulation of the stationary state of the N -species ASEP

The matrix product formulation was first used to solve the TASEP on a lattice of length L with open boundary conditions [11]. It was then extended to the 2-ASEP on the ring \mathbb{Z}_L [20], which is our starting point. In this case, the site variable is $\tau_l = 0, 1, 2$ which implies that site l is empty, contains a first-class particle or contains a second-class particle, respectively. For the 2-ASEP the matrix product formulation consists of writing the weight of a given configuration as a trace of a product of L matrices corresponding to the classes of particles on the different sites. In the matrix product formulation [20] it has been proved that the stationary measure may be written as

$$P(\vec{\tau}) = \frac{1}{Z(P_0, P_1, P_2)} W(\vec{\tau}), \quad (7)$$

where the weight of the configuration is given by

$$W(\vec{\tau}) = \text{Tr} \left[\prod_{l=1}^L X_{\tau_l} \right] \quad (8)$$

and Tr means the trace of the product of matrices X_{τ_l} . The normalization $Z(P_0, P_1, P_2)$ is chosen so that the sum of all the probabilities is equal to 1, i.e., $Z(P_0, P_1, P_2)$ is the sum of weights for configurations with the correct numbers P_0, P_1, P_2 of holes and each species of particles. The matrices X_{τ_l} are given by

$$X_0 = E, \quad X_1 = D, \quad X_2 = A, \quad (9)$$

that is, if the site is empty we write a matrix E ; if the site contains a first-class particle we write a matrix D ; if the site contains a second-class particle we write a matrix A . The matrices D, E, A obey the algebraic rules

$$DE - qED = (1 - q)(D + E) \tag{10}$$

$$DA - qAD = (1 - q)A \tag{11}$$

$$AE - qEA = (1 - q)A. \tag{12}$$

The only remaining condition to satisfy is that representations of this algebra may be found which give well-defined values for the traces appearing in (7). This may be achieved for $q < 1$ as follows. Let A be the diagonal, semi-infinite matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & q & 0 & 0 & \ddots \\ 0 & 0 & q^2 & 0 & \ddots \\ 0 & 0 & 0 & q^3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{13}$$

Then D, E may be chosen to be bidiagonal semi-infinite matrices:

$$D = \begin{pmatrix} 1 & \sqrt{c_0} & 0 & 0 & \dots \\ 0 & 1 & \sqrt{c_1} & 0 & \ddots \\ 0 & 0 & 1 & \sqrt{c_2} & \ddots \\ 0 & 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{14}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \sqrt{c_0} & 1 & 0 & 0 & \ddots \\ 0 & \sqrt{c_1} & 1 & 0 & \ddots \\ 0 & 0 & \sqrt{c_2} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{15}$$

where

$$c_n = 1 - q^{n+1}. \tag{16}$$

We note that, as A has a geometric series for diagonal, the trace of A times a finite product of D and E matrices is not divergent. See [3] for further representations of (10)–(11).

In principle, explicit formulae for the weights of each configuration may be obtained, either by using the algebraic rules or taking an advantage of an explicit representation of the matrices such as (13)–(15) (see [20] or the review [3] for details of how these calculations are performed).

In what follows, it will be useful to consider the matrices δ, ϵ defined by

$$\delta = D - \mathbb{1} \tag{17}$$

$$\epsilon = E - \mathbb{1}, \tag{18}$$

which verify the algebraic relations

$$\delta\epsilon - q\epsilon\delta = (1 - q)\mathbb{1} \tag{19}$$

$$\delta A = qA\delta \tag{20}$$

$$A\epsilon = q\epsilon A. \tag{21}$$

The first equation (19) is the q -deformed harmonic oscillator algebra (see [15] for an introduction). Writing δ, ϵ out explicitly, we have

$$\delta = \begin{pmatrix} 0 & \sqrt{c_0} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{c_1} & 0 & \ddots \\ 0 & 0 & 0 & \sqrt{c_2} & \ddots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{22}$$

$$\epsilon = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{c_0} & 0 & 0 & 0 & \ddots \\ 0 & \sqrt{c_1} & 0 & 0 & \ddots \\ 0 & 0 & \sqrt{c_2} & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{23}$$

In [28], the matrix product formulation of the stationary state of the N -TASEP was presented; here we present the generalization to the N -ASEP and give complete proofs that the solution indeed satisfies the stationarity condition.

We first fix our notation. We denote the configuration of the system $\{\tau_1, \dots, \tau_L\}$ by $\vec{\tau}$. We use $X_{\tau_l}^{(N)}$ to denote the matrix associated with the state τ_l of site l in a system containing N species of particles. Thus, the subscript $\tau_l = 0, 1, \dots, N$, indicates to which species of particle the matrix is associated, and the superscript (N) indicates that there are N species of particles in the system. This is required because the matrix corresponding to a species will vary according to the total number of species in the system. The stationary probabilities become in the matrix product formulation

$$P(\vec{\tau}) = \frac{1}{Z(P_0, \dots, P_N)} \text{Tr} [X_{\tau_1}^{(N)} X_{\tau_2}^{(N)} \dots X_{\tau_L}^{(N)}]. \tag{24}$$

The matrices $X_{\tau_l}^{(N)}$ are given by

$$X_J^{(N)} = \sum_{M=0}^{N-1} a_{JM}^{(N)} \otimes X_M^{(N-1)} \quad \text{for } 0 \leq J \leq N \tag{25}$$

with

$$X_0^{(1)} = X_1^{(1)} = 1. \tag{26}$$

Note that we use a slightly different notation to [28] for the TASEP; with our notation, some of the $a_{JM}^{(N)}$ will be equal to zero.

The matrices $a_{JM}^{(N)}$ are given by

$$a_{JM}^{(N)} = 0 \quad \text{for } 0 < M < J \tag{27}$$

$$a_{JM}^{(N)} = A^{\otimes(J-1)} \otimes \delta \otimes \mathbb{1}^{\otimes(M-J-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(N-M-1)} \quad \text{for } 0 < J < M \leq N - 1 \quad (28)$$

$$a_{0M}^{(N)} = \mathbb{1}^{\otimes(M-1)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(N-M-1)} \quad \text{for } 0 < M \leq N - 1 \quad (29)$$

$$a_{JJ}^{(N)} = A^{\otimes(J-1)} \otimes \mathbb{1}^{\otimes(N-J)} \quad \text{for } 0 < J \leq N - 1 \quad (30)$$

$$a_{00}^{(N)} = \mathbb{1}^{\otimes(N-1)} \quad \text{for } 0 < J < M \leq N - 1 \quad (31)$$

$$a_{J0}^{(N)} = A^{\otimes(J-1)} \otimes \delta \otimes \mathbb{1}^{\otimes(N-J-1)} \quad \text{for } 0 < J \leq N - 1 \quad (32)$$

$$a_{N0}^{(N)} = A^{\otimes(N-1)}. \quad (33)$$

Thus the matrices $X_K^{(N)}$ at level N are composed of tensor products of $\dim_N = \binom{N}{2}$ fundamental matrices δ, ϵ, A or $\mathbb{1}$. For $N = 1, \dim_N = 0$ which implies the use of scalars for the single-species case (26) (i.e. all configurations are equally likely). For $N = 2, \dim_N = 1$, which implies the use of matrices as given by (13)–(15). Let us now check that the 2-ASEP matrices are recovered. We find from equations (27), (29), (30) and (33) that $a_{00}^{(2)} = \mathbb{1}, a_{01}^{(2)} = \epsilon, a_{10}^{(2)} = \delta, a_{11}^{(2)} = \mathbb{1}$ and $a_{20}^{(2)} = A$. Then, from the recursion relation (25) we deduce that

$$X_0^{(2)} = a_{00}^{(2)} + a_{01}^{(2)} = \mathbb{1} + \epsilon = E, \quad X_1^{(2)} = a_{10}^{(2)} + a_{11}^{(2)} = \mathbb{1} + \delta = D, \quad (34)$$

$$\text{and } X_2^{(2)} = a_{20}^{(2)} = A, \quad (35)$$

as expected.

Let us now write out the case $N = 3$. Since $\dim_3 = 3$, the $X_K^{(3)}$ matrices are built with tensor products of three fundamental matrices. From the definitions (27)–(33), we have $a_{00}^{(3)} = \mathbb{1} \otimes \mathbb{1}, a_{01}^{(3)} = \epsilon \otimes \mathbb{1}, a_{02}^{(3)} = \mathbb{1} \otimes \epsilon, a_{10}^{(3)} = \delta \otimes \mathbb{1}, a_{11}^{(3)} = \mathbb{1} \otimes \mathbb{1}, a_{12}^{(3)} = \delta \otimes \epsilon, a_{20}^{(3)} = A \otimes \delta, a_{22}^{(3)} = A \otimes \mathbb{1}$ and $a_{30}^{(3)} = A \otimes A$. Using the recursion relation (25) and the results $X_0^{(2)} = E, X_1^{(2)} = D, X_2^{(2)} = A$, we obtain

$$\begin{aligned} X_0^{(3)} &= a_{00}^{(3)} \otimes X_0^{(2)} + a_{01}^{(3)} \otimes X_1^{(2)} + a_{02}^{(3)} \otimes X_2^{(2)} \\ &= \mathbb{1} \otimes \mathbb{1} \otimes E + \epsilon \otimes \mathbb{1} \otimes D + \mathbb{1} \otimes \epsilon \otimes A \end{aligned} \quad (36)$$

$$\begin{aligned} X_1^{(3)} &= a_{10}^{(3)} \otimes X_0^{(2)} + a_{11}^{(3)} \otimes X_1^{(2)} + a_{12}^{(3)} \otimes X_2^{(2)} \\ &= \delta \otimes \mathbb{1} \otimes E + \mathbb{1} \otimes \mathbb{1} \otimes D + \delta \otimes \epsilon \otimes A \end{aligned} \quad (37)$$

$$X_2^{(3)} = a_{20}^{(3)} \otimes X_0^{(2)} + a_{22}^{(3)} \otimes X_2^{(2)} = A \otimes \delta \otimes E + A \otimes \mathbb{1} \otimes A \quad (38)$$

$$X_3^{(3)} = a_{30}^{(3)} \otimes X_0^{(2)} = A \otimes A \otimes E. \quad (39)$$

We note in this example that the matrices $X_K^{(N)}$ have the same expression in terms of δ, ϵ and A as for the TASEP [28]; the only difference lies in the deformation by q of the algebra (19)–(21) between δ, ϵ and A .

In later sections, we shall establish the algebraic rules, satisfied by the matrices $X_{\vec{v}}^{(N)}$, which generalize the quadratic algebra (10)–(11) of the $N = 2$ case. First, we discuss how the recursive structure (25) allows the stationary weights for an N -species system to be written in terms of those for an $(N - 1)$ -species system.

4. Transfer matrix interpretation of the matrix product representation

In this section, we show that the matrix product representation (24) can be conveniently rewritten in terms of a transfer matrix acting on the configuration space.

4.1. Definition of the transfer matrix

We begin by noting that (24) and (25) define the stationary weights for a system with N species of particles recursively in terms of the stationary weights for a system with $N - 1$ species:

$$\begin{aligned} \text{Tr}[X_{j_1}^{(N)} \dots X_{j_L}^{(N)}] &= \sum_{i_1, \dots, i_L=0}^{N-1} \text{Tr}[(a_{j_1 i_1}^{(N)} \otimes X_{i_1}^{(N-1)}) \dots (a_{j_L i_L}^{(N)} \otimes X_{i_L}^{(N-1)})] \\ &= \sum_{\vec{i}} \text{Tr}[(a_{j_1 i_1}^{(N)} \dots a_{j_L i_L}^{(N)}) \otimes (X_{i_1}^{(N-1)} \dots X_{i_L}^{(N-1)})] \\ &= \sum_{\vec{i}} \text{Tr}[a_{j_1 i_1}^{(N)} \dots a_{j_L i_L}^{(N)}] \text{Tr}[X_{i_1}^{(N-1)} \dots X_{i_L}^{(N-1)}]. \end{aligned} \quad (40)$$

We use the notation $\vec{j} \equiv (j_1, \dots, j_L)$ and $\vec{i} \equiv (i_1, \dots, i_L)$, where \vec{j} is a configuration of a system with N species of particles, and \vec{i} is a configuration of a system with $(N - 1)$ species of particles. Each j_l can take values from 0 to N , and each i_l can take values from 0 to $N - 1$. The sum in (40) is over all configurations \vec{i} with $(N - 1)$ species.

Let us now introduce a notation for the configuration space which will be of use in the sequel. We call $V^{(N)}$ the $(N + 1)$ -dimensional vector space corresponding to the $N + 1$ possible states of a site for a system with N species. If we do not specify the number of particles of each species, the total configuration space of the system is $V_L^{(N)} \equiv (V^{(N)})^{\otimes L}$. We denote a vector in this space corresponding to configuration $\vec{j} = (j_1, \dots, j_L)$ by $|\vec{j}\rangle$. The set of all $(N + 1)^L$ possible configuration vectors $|\vec{j}\rangle$ forms a basis of $V_L^{(N)}$. We denote the steady-state eigenvector for a system containing N species of particles by $|N\rangle$ where

$$|N\rangle = \sum_{\vec{j}} W(\vec{j}) |\vec{j}\rangle. \quad (41)$$

The stationary state weights $W(\vec{j})$ (8) are then given by

$$W(\vec{j}) = \langle \vec{j} | N \rangle. \quad (42)$$

Using this notation we can write relation (40) in the form

$$\langle \vec{j} | N \rangle = \sum_{\vec{i}} \langle \vec{j} | T_L^{(N)} | \vec{i} \rangle \langle \vec{i} | N - 1 \rangle, \quad (43)$$

where $T_L^{(N)}$ is the *transfer matrix* for a system with N species. The matrix element $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ can be thought of as representing a transition from configuration \vec{i} in $V_L^{(N-1)}$ to configuration \vec{j} in $V_L^{(N)}$. (This transition is, of course, not the same as a dynamical transition given by the Markov matrix.) The transfer matrix is used to express the stationary weights for a system with N species linearly in terms of the weights for a system with $N - 1$ species. We identify from (40) the elements of the transfer matrix $T_L^{(N)}$ as

$$\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle \equiv \text{Tr}[a_{j_1 i_1}^{(N)} \dots a_{j_L i_L}^{(N)}]. \quad (44)$$

We can write relation (43) more simply as

$$|N\rangle = T_L^{(N)} |N - 1\rangle, \quad (45)$$

and iterating we obtain

$$|N\rangle = T_L^{(N)} \dots T_L^{(2)} |1\rangle, \quad (46)$$

where the eigenvector of the system with only one species $|1\rangle$ is such that each configuration has the same weight.

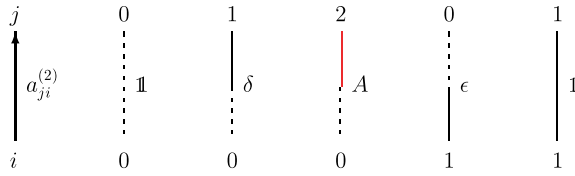


Figure 1. Allowed local transitions in the transfer matrix $T_L^{(2)}$ between a particle of class i (lower row, $i = 0$ or $i = 1$) and a particle of class j (upper row, $j = 0$, $j = 1$ or $j = 2$) at the same site. A particle of class 0 is represented by a vertical dashed line, a particle of class 1 by a full line and a particle of class 2 by a double line. The corresponding local tensor $a_{ji}^{(2)}$ appearing in the transfer matrix element is also shown.

Let us now formalize the mathematical structure of $T_L^{(N)}$. The transfer matrix $T_L^{(N)}$ is a $(N + 1)^L \times N^L$ rectangular matrix which takes a vector in $V_L^{(N-1)}$ and sends it to in $V_L^{(N)}$. It is expressed as a trace of a product of the local tensors $a_{ji}^{(N)}$. From (27) to (33), we observe that these $a_{ji}^{(N)}$ are themselves tensorial products of elements of the set $\mathcal{F} = \{\delta, \epsilon, A, \mathbb{1}\}$. Thus, the building blocks of the transfer matrix and indeed the matrices $X_j^{(N)}$ are the four infinite matrices of the set \mathcal{F} . We will call \mathcal{A} the infinite dimensional space on which the elements of \mathcal{F} act. The matrices $a_{JM}^{(N)}$ act on the auxiliary space $\mathcal{A}^{(N)} \equiv \mathcal{A}^{\otimes(N-1)}$. They can be seen as the element JM ($0 \leq J \leq N$ and $0 \leq M \leq N - 1$) of a rectangular $(N + 1) \times N$ matrix $a^{(N)}$. For example,

$$a^{(2)} = \begin{pmatrix} \mathbb{1} & \epsilon \\ \delta & \mathbb{1} \\ A & 0 \end{pmatrix} \quad \text{and} \quad a^{(3)} = \begin{pmatrix} \mathbb{1} \otimes \mathbb{1} & \epsilon \otimes \mathbb{1} & \mathbb{1} \otimes \epsilon \\ \delta \otimes \mathbb{1} & \mathbb{1} \otimes \mathbb{1} & \delta \otimes \epsilon \\ A \otimes \delta & 0 & A \otimes \mathbb{1} \\ A \otimes A & 0 & 0 \end{pmatrix}. \quad (47)$$

4.2. Interpretation of the $T_L^{(N)}$ matrices

In this subsection, we give an interpretation of the transfer matrices $T_L^{(N)}$ in terms of forbidden and allowed transitions between configurations of particles. Forbidden transitions correspond to the matrix elements of $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ that are equal to zero. We prove in appendix B that the only nonzero matrix elements $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ between an initial configuration \vec{i} of particles of species between 0 and $N - 1$, and a final configuration \vec{j} of particles of species between 0 and N are characterized by the following rules:

- at each site, a hole in the initial configuration \vec{i} can remain a hole or become a particle of any class between 1 and N in the final configuration \vec{j} ,
- at each site, a particle of class i between 1 and $N - 1$ can become either a hole or a particle of class j between 1 and i (the class can only decrease),
- there is global conservation of the number of particles of each class between 1 and $N - 1$.

These rules are proven in appendix B. In particular, we note that the number of holes decreases (or stays the same) whereas the number of particles of class N can only increase from none. Thus, the only way to create a particle of class N is to remove a hole and create a class N particle instead. The allowed local transitions for $N = 2$ and $N = 3$ along with the corresponding local tensors $a_{ji}^{(N)}$ appearing in the transfer matrix element are illustrated in figures 1 and 2.

To illustrate the utility of the transfer matrix, we work out some simple examples. First we consider the configuration $(2, 1, 0)$ for $L = 3$ and $N = 2$. According to the rules for

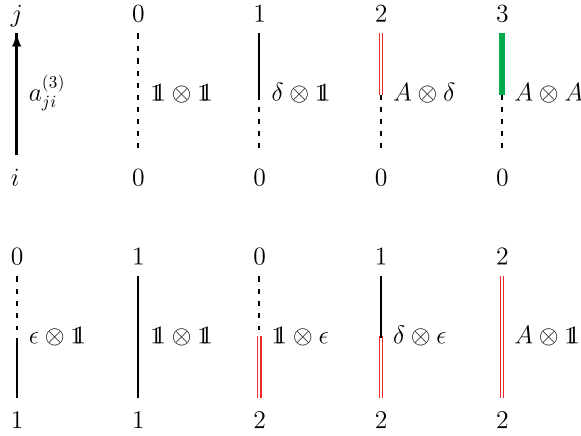


Figure 2. Allowed local transitions in the transfer matrix $T_L^{(3)}$ between a particle of class i (lower row, $i = 0, i = 1$ or $i = 2$) and a particle of class j (upper row, $j = 0, j = 1, j = 2$ or $j = 3$) at the same site. A particle of class 0 is represented by a vertical dashed line, a particle of class 1 by a full line, a particle of class 2 by a double line and a particle of class 3 by a triple line. The corresponding local tensor $a_{ji}^{(3)}$ appearing in the transfer matrix element is also shown.

nonzero elements of the transfer matrix in the beginning of this subsection, the 2 at site 1 must have come from a 0, the 1 at site 2 could have come from a 1 or 0, the 0 at site 3 could have come from a 1 or 0. Then the global constraint of a conserved number of 1 implies that the only one-species configurations which have transitions to $(2, 1, 0)$ are $(0, 1, 0)$ and $(0, 0, 1)$. Using figure 1 to construct the transfer matrix elements from the local tensors $a_{ji}^{(N)}$, we find

$$W(2, 1, 0) = W(0, 1, 0) \text{Tr} [a_{20}^{(2)} a_{11}^{(2)} a_{00}^{(2)}] + W(0, 0, 1) \text{Tr} [a_{20}^{(2)} a_{10}^{(2)} a_{01}^{(2)}] \quad (48)$$

$$= \text{Tr}[A] + \text{Tr}[A\delta\epsilon], \quad (49)$$

where we have set $W(0, 1, 0) = W(0, 0, 1) = 1$ (uniform measure for $N = 1$). This is to be compared with the known expression given by the matrices (9):

$$W(2, 1, 0) = \text{Tr}[ADE] = \text{Tr}[A(\mathbb{1} + \delta)(\mathbb{1} + \epsilon)] = \text{Tr}[A + A\delta\epsilon], \quad (50)$$

where we have used the property that δ and ϵ must appear in equal numbers to have a nonzero trace.

As another example we consider the configuration $(3, 2, 1, 0)$ for $L = 4$ and $N = 3$. According to the rules for nonzero elements of the transfer matrix: the 3 at site 1 must have come from a 0; the 2 at site 2 could have come from a 2 or 0; the 1 at site 3 could have come from a 2,1 or 0; and the 0 at site 4 could have come from a 2,1 or 0. Then the global constraints of conserved numbers of 1 and 2 imply that the only two-species configurations which have transitions to $(3, 2, 1, 0)$ are $(0, 2, 1, 0)$, $(0, 2, 0, 1)$, $(0, 0, 2, 1)$, $(0, 0, 1, 2)$. Using figure 2 to construct the transfer matrix elements yields

$$\begin{aligned} W(3, 2, 1, 0) &= W(0, 2, 1, 0) \text{Tr} [a_{30}^{(3)} a_{22}^{(3)} a_{11}^{(3)} a_{00}^{(3)}] + W(0, 2, 0, 1) \text{Tr} [a_{30}^{(3)} a_{22}^{(3)} a_{10}^{(3)} a_{01}^{(3)}] \\ &\quad + W(0, 0, 2, 1) \text{Tr} [a_{30}^{(3)} a_{20}^{(3)} a_{12}^{(3)} a_{01}^{(3)}] + W(0, 0, 1, 2) \text{Tr} [a_{30}^{(3)} a_{22}^{(3)} a_{11}^{(3)} a_{02}^{(3)}] \\ &= W(0, 2, 1, 0) \text{Tr}[A^2] \text{Tr}[A] + W(0, 2, 0, 1) \text{Tr}[A^2\delta\epsilon] \text{Tr}[A] \\ &\quad + W(0, 0, 2, 1) \text{Tr}[A^2\delta\epsilon] \text{Tr}[A\delta\epsilon] + W(0, 0, 1, 2) \text{Tr}[A^2] \text{Tr}[A\delta\epsilon]. \end{aligned} \quad (51)$$

We used the fact that the trace of a tensorial product is equal to the product of the traces.

4.3. Finiteness of the trace operation

We have so far avoided the important question of the finiteness of the matrix product representation (24). Using previous results of this section and the appendices, we now prove that if there is at least one particle of each species, then expression (24) for the stationary measure of the N -species ASEP is finite. (In the case where there are zero particles of some species, this species can be removed from the problem by studying the corresponding system with $N - 1$ species instead of N .) We will focus on the case $N = 3$. For any configuration \vec{k} with three species, the stationary weight is given by

$$W(\vec{k}) = \langle \vec{k} | 3 \rangle = \sum_{\vec{j}} \sum_{\vec{i}} \langle \vec{k} | T_L^{(3)} | \vec{j} \rangle \langle \vec{j} | T_L^{(2)} | \vec{i} \rangle \langle \vec{i} | 1 \rangle, \quad (52)$$

where \vec{i} is a configuration with one species and \vec{j} is a configuration with two species. Let us assume that there is at least one particle of each class in \vec{k} . Then, as $T_L^{(3)}$ conserves the number of particles of class 2, the configurations \vec{j} that give a nonzero contribution to $\langle \vec{k} | 3 \rangle$ are such that \vec{j} has also at least one particle of class 2. From the characterization of the nonzero elements of the transfer matrix, we must have that both $i_l = 0$, $j_l = 2$ and $j_{l'} = 0$, $k_{l'} = 3$ at some sites l and l' between 1 and L so that \vec{i} and \vec{j} give a nonzero contribution to $\langle \vec{k} | 3 \rangle$. We see, using the expression of the transfer matrix elements (44) and the form (33) of $a_{N0}^{(N)}$ for $N = 2$ and $N = 3$, that there will be at least one A in each trace of $\langle \vec{j} | T_L^{(2)} | \vec{i} \rangle$ and $\langle \vec{k} | T_L^{(3)} | \vec{j} \rangle$ contributing to $\langle \vec{k} | 3 \rangle$. But one can calculate explicitly traces of products of elements of \mathcal{F} when there is at least one A in the product, and these traces are finite (see appendix A). This proves that $\langle \vec{k} | 3 \rangle$ is finite. The extension to arbitrary N of this proof of the finiteness of the matrix product representation (24) is straightforward.

4.4. Relation with Ferrari–Martin’s construction for the TASEP

For the totally asymmetric case ($q = 0$), the transfer matrices $T_L^{(N)}$ encode Ferrari–Martin’s multiline construction of the stationary weights [27]. We will focus on the two-species stationary eigenstate constructed by $T_L^{(2)}$. We will see that the set of pairs of configurations \vec{i} and \vec{j} for which $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle \neq 0$ is smaller than in the case $q \neq 0$. In order to know for which pairs of configurations \vec{i} and \vec{j} the matrix element $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ is equal to zero, we refer to representation (B.4) of appendix B which expresses an element of the transfer matrix as a product of $N - 1$ traces of products of L fundamental matrices $\{\delta, \epsilon, A, \mathbb{1}\}$. Therefore, we need to know which products of elements of $\{\delta, \epsilon, A, \mathbb{1}\}$ have a trace equal to zero. For the case of the TASEP, i.e. $q = 0$, the algebra (19), (20), (21) between δ , ϵ and A reduces to

$$\delta\epsilon = \mathbb{1} \quad (53)$$

$$\delta A = 0 \quad (54)$$

$$A\epsilon = 0. \quad (55)$$

One can see that for any product of l elements from $\{\delta, \epsilon\}$, $w = w_1 \dots w_l$, $AwA \neq 0$ if and only if both of the following conditions are true:

- there are as many δ and ϵ in w ,
- for all m between 1 and l , there is at least as many ϵ as δ in $w_m \dots w_l$.

In that case, $AwA = AA = A$. For example, $A\delta\delta\epsilon\epsilon\delta\epsilon A = A$ but $A\delta\epsilon\epsilon\delta\delta\epsilon A = 0$ because of an excess of δ to the right. Then, each of the traces of products of elements from $\{\delta, \epsilon, A, \mathbb{1}\}$

appearing in the transfer matrix element (B.4) will either be 0 or 1, and consequently each transfer matrix element is either 0 or 1. Then the transfer matrix relation (43) becomes an expression for the weight of a configuration in a system of N species as a sum of weights for ‘ancestor’ configurations of a system of $N - 1$ species. The rules for selecting these ancestor configurations are precisely those given by Ferrari and Martin.

We illustrate the equivalence for $q = 0$ of the transfer matrix relation (43) with the Ferrari and Martin algorithm in the case $N = 2$, studied by Angel [29]. In this case (see figure 1) an ϵ in the transfer matrix element corresponds to a particle of class 1 changing into a hole; a δ corresponds to a hole changing into a particle of class 1 and an A to a hole changing into a particle of class 2. Therefore, for $q = 0$, $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle \neq 0$ if and only if the configurations \vec{i} and \vec{j} are such that one can go from \vec{i} to \vec{j} by creating particles of class 2 at some of the unoccupied sites and by moving particles of class 1 only to the left, forbidding them to cross class 2 particles: this is precisely the pushing procedure of Angel for two classes of particles which is a particular (2-line) case of the Ferrari–Martin N -line algorithm (see [28]).

As a simple example of the distinction between the $q = 0$ and $q \neq 0$ cases let us consider the configuration (0, 1, 2) for $L = 3$ and $N = 2$. Constructing the transfer matrix elements as before, we find

$$W(0, 1, 2) = W(0, 1, 0) \text{Tr} [a_{00}^{(2)} a_{11}^{(2)} a_{20}^{(2)}] + W(1, 0, 0) \text{Tr} [a_{01}^{(2)} a_{10}^{(2)} a_{20}^{(2)}] = \text{Tr}[A] + \text{Tr}[\epsilon \delta A], \tag{56}$$

where we have set $W(0, 1, 0) = W(1, 0, 0)$. In the case $q = 0$, $\text{Tr}[\epsilon \delta A] = 0$, and there is only one contribution to $W(0, 1, 2)$. This concurs with the pushing procedure for this example which results in just one ancestor configuration (0, 1, 0). However, for $q \neq 0$, the second term in (56) does contribute and using the result of appendix A one finds

$$\text{Tr}[\epsilon \delta A] = \frac{q}{1+q} \text{Tr}[A].$$

A further important observation for the 2-TASEP is that the matrix elements $\langle \vec{j} | T_L^{(2)} | \vec{i} \rangle$ ‘decouple on the particles of class 2’, that is, each element of the transfer matrix $T_L^{(2)}$ factorizes as a product over all the pairs of two consecutive particles of class 2 in \vec{j} of terms depending only on the holes and first-class particles in \vec{i} and \vec{j} between the particles of class 2. This is because for $q = 0$ the matrix A is a projector (13). However, this factorization property does not hold anymore for the general case $q \neq 0$ as (13) is no longer a projector. This fact makes it more difficult to find a combinatorial interpretation of the transfer matrix such as Ferrari–Martin multiline construction since the hops of the particles are less restricted than for the TASEP.

5. Quadratic algebra for the $a_{JM}^{(N)}$

From the algebra for δ, ϵ and A (19)–(21), and the explicit form of the $a_{JM}^{(N)}$ (27)–(33), many quadratic relations for the $a_{JM}^{(N)}$ can be deduced. In the proof of the matrix product representation, to be presented in section 6, we will need two kinds of these relations: symmetries of a quadratic function of the $a_{JM}^{(N)}$ under the exchange of the second indices of the $a_{JM}^{(N)}$ and relations allowing us to commute two $a_{JM}^{(N)}$.

5.1. Symmetry relations

We have the following relation between commutators:

$$[a_{JM}^{(N)}, a_{KM'}^{(N)}] = [a_{JM'}^{(N)}, a_{KM}^{(N)}] \quad \text{for } J \neq 0 \quad \text{and} \quad K \neq 0, \tag{57}$$

which indicates that $[a_{JM}^{(N)}, a_{KM'}^{(N)}]$ is symmetric under the exchange of M and M' for all M and M' . For some values of J, K, M and M' , we have the stronger properties:

$$a_{JM}^{(N)}a_{KM'}^{(N)} = a_{JM'}^{(N)}a_{KM}^{(N)} \quad \text{for all } J, K, \text{ and } M, M' \in \{0\} \cup [\max(J + 1, K + 1), N - 1] \quad (58)$$

$$a_{JM}^{(N)}a_{JM'}^{(N)} = a_{JM'}^{(N)}a_{JM}^{(N)} \quad \text{for all } J, M \text{ and } M', \quad (59)$$

which are also symmetries under the exchange between M and M' .

5.2. Commutation relations

In the following, we will also need to exchange $a_{JM}^{(N)}$ and $a_{KM'}^{(N)}$. If $J = K$, the symmetry relation (59) can also be seen as a commutation relation for all values of J, M and M' . For $J \neq K$, the exact form of the commutation relation will depend on M and M' . In the case $0 < J < K$ we partition the ensemble of couples (M, M') for which $a_{JM}^{(N)}$ and $a_{KM'}^{(N)}$ are defined and nonzero (that is $M \in \{0\} \cup [J, N - 1]$ and $M' \in \{0\} \cup [K, N - 1]$) into 12 sectors as

	$M = 0$	$J \leq M < K$	$M = K$	$K < M \leq N - 1$	
$M' = 0$	1	4	7	10	(60)
$M' = K$	2	5	8	11	
$K < M' \leq N - 1$	3	6	9	12.	

Then, we have the following commutation relations between $a_{JM}^{(N)}$ and $a_{KM'}^{(N)}$ for $0 < J < K$:

$$a_{JM}^{(N)}a_{KM'}^{(N)} = qa_{KM'}^{(N)}a_{JM}^{(N)} \quad \text{in sectors 1, 2, 3, 8, 10, 11 and 12} \quad (61)$$

$$a_{JM}^{(N)}a_{KM'}^{(N)} = a_{KM'}^{(N)}a_{JM}^{(N)} \quad \text{in sectors 4, 5, 6} \quad (62)$$

$$a_{JM}^{(N)}a_{KM'}^{(N)} = a_{KM'}^{(N)}a_{JM}^{(N)} - (1 - q)a_{KM}^{(N)}a_{JM'}^{(N)} \quad \text{in sectors 7 and 9.} \quad (63)$$

In the case $K = 0 < J$, the commutation relations take a similar form (C.2)–(C.4).

6. Proof of the matrix product representation

In this section, we prove that the matrix product expression (24) gives the stationary-state eigenvector of the Markov matrix.

6.1. ‘Hat’ matrices

The Markov matrix of the system with N classes of particles can be written in terms of the local $(N + 1)^2$ by $(N + 1)^2$ matrices $M_{k,k+1}^{(N)}$ which encode the rates at which the particles hop between site k and $k + 1$:

$$M = \sum_{k=1}^L \mathbb{1}^{\otimes(k-1)} \otimes M_{k,k+1}^{(N)} \otimes \mathbb{1}^{\otimes(L-k-1)}. \quad (64)$$

For a model for which the rates do not depend on the site such as that we are discussing, the local matrices do not depend on the site: $M_{k,k+1}^{(N)} \equiv M_{\text{loc}}^{(N)}$. For example, for $N = 2$, in the basis (11, 12, 10, 21, 22, 20, 01, 02, 00)

$$M_{\text{loc}}^{(2)} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & q & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & q & \cdot & \cdot \\ \cdot & 1 & \cdot & -q & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & q & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -q & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -q & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (65)$$

The dots represent matrix elements equal to zero. The matrix product expression (24) gives the stationary state eigenvector if for all configuration \vec{j} with a particle of class j_k at site k

$$\sum_{\vec{i}} M_{\vec{j},\vec{i}} \text{Tr} [X_{i_1}^{(N)} \cdots X_{i_L}^{(N)}] = 0, \quad (66)$$

or in terms of the local jump matrix

$$\sum_{k=1}^L \sum_{i_k, i_{k+1}=0}^N (M_{\text{loc}}^{(N)})_{j_k j_{k+1}, i_k i_{k+1}} \text{Tr} [X_{j_1}^{(N)} \cdots X_{j_{k-1}}^{(N)} X_{i_k}^{(N)} X_{i_{k+1}}^{(N)} X_{j_{k+2}}^{(N)} \cdots X_{j_L}^{(N)}] = 0. \quad (67)$$

This equation will be satisfied if there exist some additional ‘hat’ matrices [30, 31] $\hat{X}_0^{(N)}, \dots, \hat{X}_N^{(N)}$ such that

$$\sum_{i,i'=0}^N (M_{\text{loc}}^{(N)})_{jj',ii'} X_i^{(N)} X_{i'}^{(N)} = X_j^{(N)} \hat{X}_{j'}^{(N)} - \hat{X}_j^{(N)} X_{j'}^{(N)}, \quad (68)$$

leading to a cancellation of all the terms two by two. Knowing the form of the local matrix $M_{\text{loc}}^{(N)}$, the previous equation (68) can be rewritten as

$$\text{for } 0 \leq J \leq N [X_J^{(N)}, \hat{X}_J^{(N)}] = 0, \quad (69)$$

$$\text{for } 0 < J < K \leq N \text{ or } 0 = K < J \leq N \\ X_J^{(N)} X_K^{(N)} - q X_K^{(N)} X_J^{(N)} = \hat{X}_J^{(N)} X_K^{(N)} - X_J^{(N)} \hat{X}_K^{(N)} \quad (70)$$

$$= X_K^{(N)} \hat{X}_J^{(N)} - \hat{X}_K^{(N)} X_J^{(N)}. \quad (71)$$

Again, the fact that the case $0 = K < J \leq N$ is singled out, comes from our choice to give the holes the index 0 instead of $N + 1$ as the particle hierarchy would have required. Subtracting (70) from (71) we also have the additional relation valid for all values of J and K :

$$[X_J^{(N)}, \hat{X}_K^{(N)}] + [X_K^{(N)}, \hat{X}_J^{(N)}] = 0. \quad (72)$$

This equation holds even for $J = K$ because of equation (69). It tells us that $[X_J^{(N)}, \hat{X}_K^{(N)}]$ must be antisymmetric under the exchange between J and K .

In the following subsection we will prove by induction on N that the following $\hat{X}^{(N)}$ matrices and the $X^{(N)}$ matrices defined in equation (25) verify equations (69) and (70):

$$\hat{X}_0^{(N)} = -(1 - q) X_0^{(N)} + \sum_{M=0}^{N-1} a_{0M}^{(N)} \otimes \hat{X}_M^{(N-1)} \quad (73)$$

$$\hat{X}_J^{(N)} = \sum_{M=0}^{N-1} a_{JM}^{(N)} \otimes \hat{X}_M^{(N-1)} \quad \text{for } 1 \leq J \leq N, \quad (74)$$

with $\hat{X}_0^{(1)} = 1$ and $\hat{X}_1^{(1)} = q$. The $a_{JM}^{(N)}$ are still given by equations (27)–(33). In contrast to the $X_J^{(N)}$ matrices (25), the expression for the $\hat{X}_J^{(N)}$ in terms of the $a_{JM}^{(N)}$ is not exactly the same as for the TASEP: q enters both (73) and the value of $\hat{X}_1^{(1)}$.

We first check the simple cases $N = 1$ and $N = 2$. Using $X_0^{(1)} = 1$, $X_1^{(1)} = 1$, $\hat{X}_0^{(1)} = 1$ and $\hat{X}_1^{(1)} = q$, we see that relations (69)–(71) are trivially satisfied for $N = 1$. For $N = 2$ we find

$$\begin{aligned} \hat{X}_0^{(2)} &= -(1-q)X_0^{(2)} + a_{00}^{(2)} \otimes \hat{X}_0^{(1)} + a_{01}^{(2)} \otimes \hat{X}_1^{(1)} \\ &= -(1-q)E + q\mathbb{1} + \epsilon = qE - (1-q)\mathbb{1} \\ \hat{X}_1^{(2)} &= a_{10}^{(2)} \otimes \hat{X}_0^{(1)} + a_{11}^{(2)} \otimes \hat{X}_1^{(1)} \\ &= q\delta + \mathbb{1} = qD + (1-q)\mathbb{1} \\ \hat{X}_2^{(2)} &= a_{10}^{(2)} \otimes \hat{X}_0^{(1)} = qA. \end{aligned}$$

Clearly, because $X_0^{(2)} = E$, $X_1^{(2)} = D$ and $X_2^{(2)} = A$, the commutation relations (69) are satisfied. One can check, for example, that

$$\begin{aligned} \hat{X}_1^{(N)} X_0^{(N)} - X_1^{(N)} \hat{X}_0^{(N)} &= (qD + (1-q)\mathbb{1})E - D(qE - (1-q)\mathbb{1}) \\ &= (1-q)(D + E) \\ X_0^{(N)} \hat{X}_1^{(N)} - \hat{X}_0^{(N)} X_1^{(N)} &= E(qD + (1-q)\mathbb{1}) - (qE - (1-q)\mathbb{1})D \\ &= (1-q)(D + E), \end{aligned}$$

so that (70) and (71) are satisfied for $J = 1$, $K = 0$, and the other cases are similarly verified.

In the following subsection, we prove the three quadratic relations (69)–(71) which are sufficient for the matrix product expression (24) to give the stationary state eigenvector of the Markov matrix. For each relation, we must distinguish the case where some indices are equal to zero (which will be proved in appendix C), giving six relations to prove. The proofs all have a similar pattern. They are done by induction on N and rely heavily on the symmetry and commutation relations of section 5.

6.2. Proof of equation (69)

If $J \neq 0$ we have

$$[X_J^{(N)}, \hat{X}_J^{(N)}] = \sum_{M=0}^{N-1} \sum_{M'=0}^{N-1} [(a_{JM}^{(N)} a_{JM'}^{(N)}) \otimes (X_M^{(N-1)} \hat{X}_{M'}^{(N-1)}) - (a_{JM'}^{(N)} a_{JM}^{(N)}) \otimes (\hat{X}_{M'}^{(N-1)} X_M^{(N-1)})]. \quad (75)$$

Using the quadratic relation (59) to exchange $a_{JM'}^{(N)}$ and $a_{JM}^{(N)}$, we get

$$[X_J^{(N)}, \hat{X}_J^{(N)}] = \sum_{M=0}^{N-1} \sum_{M'=0}^{N-1} (a_{JM}^{(N)} a_{JM'}^{(N)}) \otimes [X_M^{(N-1)}, \hat{X}_{M'}^{(N-1)}]. \quad (76)$$

Using the fact that $a_{JM}^{(N)} a_{JM'}^{(N)}$ is symmetric in M and M' (59), and that from (72) $[X_M^{(N-1)}, \hat{X}_{M'}^{(N-1)}]$ is antisymmetric in M and M' by induction, we find that $[X_J^{(N)}, \hat{X}_J^{(N)}] = 0$.

For $J = 0$, we have an extra term compared to (75):

$$[X_0^{(N)}, \hat{X}_0^{(N)}] = [X_0^{(N)}, -(1-q)X_0^{(N)}] + \sum_{M=0}^{N-1} \sum_{M'=0}^{N-1} [(a_{0M}^{(N)} a_{0M'}^{(N)}) \otimes (X_M^{(N-1)} \hat{X}_{M'}^{(N-1)}) - (a_{0M'}^{(N)} a_{0M}^{(N)}) \otimes (\hat{X}_{M'}^{(N-1)} X_M^{(N-1)})]. \quad (77)$$

But this extra term, being the commutator of $X_0^{(N)}$ with itself, vanishes and (77) reduces to (76) in the case $J = 0$ giving also $[X_0^{(N)}, \hat{X}_0^{(N)}] = 0$.

In the rest of the proofs, we will use the following convention to lighten the notation: $X_M^{(N-1)}$ and $\hat{X}_M^{(N-1)}$ will be written respectively X_M and \hat{X}_M , while the $a_{JM}^{(N)}$ will just be written a_{JM} .

6.3. Proof of equation (70) ($0 < J < K$)

Let us define

$$\mathcal{A} = X_J^{(N)} X_K^{(N)} - q X_K^{(N)} X_J^{(N)} - \hat{X}_J^{(N)} X_K^{(N)} + X_J^{(N)} \hat{X}_K^{(N)}. \quad (78)$$

We want to show that $\mathcal{A} = 0$. We have

$$\mathcal{A} = \sum_{M \in \{0\} \cup [J, N-1]} \sum_{M' \in \{0\} \cup [K, N-1]} [(a_{JM} a_{KM'}) \otimes (X_M X_{M'}) - q(a_{KM'} a_{JM}) \otimes (X_{M'} X_M) - (a_{JM} a_{KM'}) \otimes (\hat{X}_M X_{M'}) + (a_{JM} a_{KM'}) \otimes (X_M \hat{X}_{M'})]. \quad (79)$$

We will cut the double sum into four parts and write $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4$, gathering sectors from the partition (60). \mathcal{A}_1 will be made of the sectors 4, 5 and 6 of (60), \mathcal{A}_2 of the sectors 1, 3, 10 and 12, \mathcal{A}_3 of the sectors 2, 7, 9 and 11, and \mathcal{A}_4 of the sector 8:

$$\mathcal{A}_1 : M \in [J, K-1] \quad \text{and} \quad M' \in \{0\} \cup [K, N-1] \quad (80)$$

$$\mathcal{A}_2 : M \in \{0\} \cup [K+1, N-1] \quad \text{and} \quad M' \in \{0\} \cup [K+1, N-1] \quad (81)$$

$$\mathcal{A}_3 : \begin{cases} M \in \{0\} \cup [K+1, N-1] & \text{and} & M' = K \\ M = K & \text{and} & M' \in \{0\} \cup [K+1, N-1] \end{cases} \quad (82)$$

$$\mathcal{A}_4 : M = K \quad \text{and} \quad M' = K. \quad (83)$$

We will now show that $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 are all equal to zero.

We begin with \mathcal{A}_1 and use the commutation relation (62). We get

$$\mathcal{A}_1 = \sum_{M=J}^{K-1} \sum_{M' \in \{0\} \cup [K, N-1]} (a_{JM} a_{KM'}) \otimes (X_M X_{M'} - q X_{M'} X_M - \hat{X}_M X_{M'} + X_M \hat{X}_{M'}). \quad (84)$$

By induction, $X_M X_{M'} - q X_{M'} X_M - \hat{X}_M X_{M'} + X_M \hat{X}_{M'}$ vanishes (70), and thus $\mathcal{A}_1 = 0$.

For \mathcal{A}_2 , using the commutation relation (61), we obtain

$$\mathcal{A}_2 = \sum_{M, M' \in \{0\} \cup [K+1, N-1]} (a_{JM} a_{KM'}) \otimes ([X_M, X_{M'}] - \hat{X}_M X_{M'} + X_M \hat{X}_{M'}). \quad (85)$$

From (58), $a_{JM} a_{KM'}$ is symmetric under the exchange of M and M' while $[X_M, X_{M'}]$ is antisymmetric, as well as $-\hat{X}_M X_{M'} + X_M \hat{X}_{M'}$ which follows from a rewriting of (72), by induction. This gives $\mathcal{A}_2 = 0$.

For \mathcal{A}_3 , using the commutation relations (61) and (63), we have

$$\begin{aligned} \mathcal{A}_3 = & \sum_{M' \in \{0\} \cup [K+1, N-1]} [(a_{JK}a_{KM'}) \otimes (X_K X_{M'} - q X_{M'} X_K - \hat{X}_K X_{M'} + X_K \hat{X}_{M'}) \\ & - q(1-q)(a_{KK}a_{JM'}) \otimes (X_{M'} X_K)] + \sum_{M \in \{0\} \cup [K+1, N-1]} (a_{JM}a_{KK}) \\ & \otimes (X_M X_K - X_K X_M - \hat{X}_M X_K + X_M \hat{X}_K). \end{aligned} \quad (86)$$

The first sum corresponds to setting $M = K$ and the second to $M' = K$. By induction, $X_K X_{M'} - q X_{M'} X_K - \hat{X}_K X_{M'} + X_K \hat{X}_{M'} = 0$ in the first sum (70). With the help of the commutation relation (61) in sectors 2 and 11 for $a_{KK}a_{JM'}$, we obtain, after renaming the dummy variable M' to M ,

$$\mathcal{A}_3 = \sum_{M \in \{0\} \cup [K+1, N-1]} (a_{JM}a_{KK}) \otimes (q X_M X_K - X_K X_M - \hat{X}_M X_K + X_M \hat{X}_K). \quad (87)$$

By induction, $q X_M X_K - X_K X_M - \hat{X}_M X_K + X_M \hat{X}_K = 0$ from (71), as only $M = 0$ and $M > K$ contribute to the sum. \mathcal{A}_3 is therefore equal to 0.

Finally, for \mathcal{A}_4 , we have

$$\mathcal{A}_4 = (a_{JK}a_{KK} - qa_{KK}a_{JK}) \otimes (X_K X_K) + (a_{JK}a_{KK}) \otimes (-\hat{X}_K X_K + X_K \hat{X}_K). \quad (88)$$

In the first term $a_{JK}a_{KK} - qa_{KK}a_{JK} = 0$ because of the commutation relation (61), and in the last term $-\hat{X}_K X_K + X_K \hat{X}_K = 0$ by induction because of (69). Thus $\mathcal{A}_4 = 0$ which proves (70) in the case $0 < J < K$.

6.4. Proof of equation (71) ($0 < J < K$)

Let us define

$$\mathcal{B} = X_J^{(N)} X_K^{(N)} - q X_K^{(N)} X_J^{(N)} - X_K^{(N)} \hat{X}_J^{(N)} + \hat{X}_K^{(N)} X_J^{(N)}. \quad (89)$$

We want to show that $\mathcal{B} = 0$. We have

$$\begin{aligned} \mathcal{B} = & \sum_{M \in \{0\} \cup [J, N-1]} \sum_{M' \in \{0\} \cup [K, N-1]} [(a_{JM}a_{KM'}) \otimes (X_M X_{M'}) - q(a_{KM'}a_{JM}) \otimes (X_{M'} X_M) \\ & - (a_{KM'}a_{JM}) \otimes (X_{M'} \hat{X}_M) + (a_{KM'}a_{JM}) \otimes (\hat{X}_{M'} X_M)]. \end{aligned} \quad (90)$$

Again, we cut the double sum into four parts and write $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$, gathering sectors from the partition (60). We use the same sectors as in the previous proof (80)–(83). At this point, we have to calculate $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and \mathcal{B}_4 . Using the same arguments as in subsection 6.3, we find that they are all equal to zero, which proves (71) in the case $0 < J < K$.

7. Conclusion

A solution for the stationary state of the multispecies TASEP was first proposed by Ferrari and Martin in [27]. Their solution was expressed in terms of numbers of configurations of a multiline queuing process. A matrix product representation of this solution was then given in [28] for the case of periodic boundary conditions, making the link with several works on one-dimensional exclusion processes in the physics literature. In this paper, we have extended this matrix product solution to the previously unsolved problem of the multispecies partially asymmetric exclusion process on a ring. In this case, there is no known analogue to Ferrari–Martin queuing construction.

The mathematical structure of the matrix product solution for the stationary state reveals several interesting features. First, the matrices are defined in a recursive fashion (see equation (25)) using auxiliary matrices $a_{ij}^{(N)}$, and are ultimately built as tensor products of the fundamental matrices $\{\delta, \epsilon, A, \mathbb{1}\}$ used in the $N = 2$ solution. Second, the matrices obey quadratic relations involving additional hat matrices and these relations generalize the quadratic algebra of the $N = 2$ case (10)–(11). Such relations have only been verified before in a few cases (see [3] section 9 for a discussion). In our solution the key to satisfying these conditions lies in the algebraic properties of the auxiliary matrices $a_{ij}^{(N)}$ which we presented in section 5.

The recursive structure of the solution allows us to construct a transfer matrix relating the stationary weight of a configuration with N species of particles to the weights of configurations with $N - 1$ species. For the case $q = 0$, the transfer matrix recovers the algorithm of Ferrari and Martin, whereas for $q \neq 0$ the structure is more complicated.

In this paper, we have not attempted to calculate physical quantities of interest, such as correlation functions. Such calculations begin with the computation of the normalization $Z(P_0, \dots, P_N)$, defined in (24), for all system sizes and particle numbers. Even for the two-species case, this computation is not easy and to our knowledge has only been carried out for the totally asymmetric case [20]. The transfer matrix construction of section 4 may provide a formalism for the computation of the normalization for partial asymmetry and general N .

There is a well-known correspondence between the matrix product representation for the one-species ASEP with open boundaries and the two-species ASEP on a ring: the matrices corresponding to the holes and to the first-class particles are the same in both cases, but the matrix corresponding to the second-class particles on the ring becomes associated with the boundaries for the open system. This correspondence should be investigated further in the case of general N species systems. Another interesting extension is the case of systems with different rates for the different classes of particles, which contains, in particular, the ABC model [32].

The multispecies asymmetric exclusion process shares with a small number of statistical physics models the significant property of being integrable. In particular, this means that its Markov matrix can be diagonalized using the (nested) Bethe ansatz. The matrix product representation of the stationary state and the Bethe ansatz [5, 8, 10, 33–39] are two of the most used techniques used to obtain exact results for the ASEP. Thanks to its rich structure; the solution of the multispecies ASEP might help to understand more precisely their relationship.

Acknowledgments

We thank Deepak Dhar and Pablo Ferrari for many useful discussions.

Appendix A. Traces of products of δ , ϵ and A

In this appendix, we show that the product of elements of the set $\mathcal{F} = \{\delta, \epsilon, A, \mathbb{1}\}$ with at least one A has a nonzero trace for $q \neq 0$ if and only if there is the same number of δ and ϵ in the product. (The case $q = 0$ has already been dealt with in section 4.4.) We use an explicit calculation of one of these traces. For $q \neq 0$ we can first use relations (20) and (21) to rearrange any product of elements of \mathcal{F} by putting all the A to the left. The result will be proportional to $\text{Tr}(A^p w)$ where w is a product of elements from the set $\{\delta, \epsilon\}$. We call r the number of δ in w and s the number of ϵ . Using again (20) and (21) to commute all the A matrices to the right, we get

$$\text{Tr}(A^p w) = q^{p(s-r)} \text{Tr}(w A^p). \tag{A.1}$$

Using the cyclicity of the trace, we see that if $p \geq 1$ and $r \neq s$, $\text{Tr}(A^p w)$ must be equal to 0.

If $p \geq 1$ but $r = s$, the algebra (19) between δ and ϵ allows one to rewrite w as a linear combination with nonnegative coefficients (for q between 0 and 1) of terms of the form $\epsilon^{r'} \delta^{r'}$ with $r' \leq r$. We want to show that $\text{Tr}(A^p \epsilon^{r'} \delta^{r'}) > 0$ in order to prove that $\text{Tr}(A^p w) \neq 0$ if $p \geq 1$ and $r = s$.

Let us define

$$f_r^{(p)} \equiv \text{Tr}(A^p \delta^r \epsilon^r). \tag{A.2}$$

We now show how $f_r^{(p)}$ can be calculated easily by recursion on r . Using first the deformed commutator (19) between δ and ϵ once, we get

$$f_{r+1}^{(p)} = \text{Tr}(A^p \delta^r (q\epsilon\delta + (1-q)\epsilon^r)) = (1-q)f_r^{(p)} + q \text{Tr}(A^p \delta^r \epsilon \delta \epsilon^r). \tag{A.3}$$

Using (19) r more times to push the rightmost δ to the right, we obtain

$$\begin{aligned} f_{r+1}^{(p)} &= \dots = (1-q)(1+q+\dots+q^k)f_r^{(p)} + q^{k+1} \text{Tr}(A^p \delta^r \epsilon^{k+1} \delta \epsilon^{r-k}) \\ &= \dots = (1-q^{r+1})f_r^{(p)} + q^{r+1} \text{Tr}(A^p \delta^r \epsilon^{r+1} \delta). \end{aligned} \tag{A.4}$$

Using equation (20) to commute the rightmost δ through all the A , we get

$$f_{r+1}^{(p)} = (1-q^{r+1})f_r^{(p)} + q^{p+r+1} f_{r+1}^{(p)}. \tag{A.5}$$

We have found the recurrence relation,

$$f_{r+1}^{(p)} = \frac{1-q^{r+1}}{1-q^{p+r+1}} f_r^{(p)}, \tag{A.6}$$

which gives

$$f_r^{(p)} = \frac{\text{Tr } A^p}{\begin{bmatrix} r+p \\ p \end{bmatrix}_q}. \tag{A.7}$$

Here $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the q -deformed binomial coefficient (see, e.g., [15]) defined as

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q [a-1]_q \dots [1]_q}{([b]_q [b-1]_q \dots [1]_q) ([a-b]_q [a-b-1]_q \dots [1]_q)} \xrightarrow{q \rightarrow 1} \binom{a}{b}, \tag{A.8}$$

the q -deformed numbers $[a]_q$ being defined by

$$[a]_q = \frac{1-q^a}{1-q} = 1+q+\dots+q^{a-1} \xrightarrow{q \rightarrow 1} a. \tag{A.9}$$

Also, using representation (13) we may evaluate

$$\text{Tr } A^p = \frac{1}{1-q^p}. \tag{A.10}$$

Thus, $f_{r'}^{(p)}$ are positive for all r' up to the factor $\text{Tr } A^p$, which means that $\text{Tr}(A^p w)$ cannot be equal to zero if $p \geq 1$ and $r = s$. We have found a necessary and sufficient condition for a product of elements from \mathcal{F} to be different from 0.

Appendix B. Proof of the characterization of the transfer matrix

We now prove the characterization of the nonzero matrix elements of the transfer matrix $T_L^{(N)}$, given in section 4.2. From (44), $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ is a trace of products of $a_{j_l i_l}^{(N)}$. Because of (27), if there exists a site l between 1 and L such that $0 < i_l < j_l$, then $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle = 0$. Thus, for all l , we must have either $i_l = 0$ or $j_l \leq i_l$ for the matrix element of $T_L^{(N)}$ to be nonzero. In terms of the particles in the configurations \vec{i} and \vec{j} , this means that a hole in \vec{i} can become a particle of any class between 0 and N by the application of the transfer matrix, and that a particle of class $i_l \geq 1$ can become a particle of class j_l only if $0 \leq j_l \leq i_l$. For example, for $N = 2$ (figure 1), only the five following transitions are allowed in the transfer matrix: $0 \rightarrow 0, 0 \rightarrow 1, 0 \rightarrow 2, 1 \rightarrow 0$ and $1 \rightarrow 1$. The transition $1 \rightarrow 2$ is forbidden. For $N = 3$ (figure 2), nine transitions are allowed: $0 \rightarrow 0, 0 \rightarrow 1, 0 \rightarrow 2, 0 \rightarrow 3, 1 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 0, 2 \rightarrow 1, 2 \rightarrow 2$. The transitions $1 \rightarrow 2, 1 \rightarrow 3$ and $2 \rightarrow 3$ are forbidden.

But this local constraint on the classes of the particles in \vec{i} and \vec{j} at each site l does not characterize completely the nonzero matrix elements of $T_L^{(N)}$: we will see that there is also a non-local constraint on \vec{i} and \vec{j} . We observe that the expression for the transfer matrix $T_L^{(N)}$ (44) can be simplified further by noting that the $a_{j_l i_l}^{(N)}$ are themselves tensorial products of $N - 1$ elements of $\mathcal{F} = \{\delta, \epsilon, A, \mathbb{1}\}$ (27)–(33). Introducing the notation

$$a_{j_l i_l}^{(N)} = a_{j_l i_l}^{(N,1)} \otimes \dots \otimes a_{j_l i_l}^{(N,N-1)}, \quad (\text{B.1})$$

we have

$$\begin{aligned} a_{j_1 i_1}^{(N)} \dots a_{j_L i_L}^{(N)} &= (a_{j_1 i_1}^{(N,1)} \otimes \dots \otimes a_{j_1 i_1}^{(N,N-1)}) \dots (a_{j_L i_L}^{(N,1)} \otimes \dots \otimes a_{j_L i_L}^{(N,N-1)}) \\ &= (a_{j_1 i_1}^{(N,1)} \dots a_{j_L i_L}^{(N,1)}) \otimes \dots \otimes (a_{j_1 i_1}^{(N,N-1)} \dots a_{j_L i_L}^{(N,N-1)}), \end{aligned} \quad (\text{B.2})$$

the $a_{j_l i_l}^{(N,k)}$ being elements of \mathcal{F} . Thus, we have for the transfer matrix element

$$\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle = \text{Tr} (a_{j_1 i_1}^{(N,1)} \dots a_{j_L i_L}^{(N,1)}) \times \dots \times \text{Tr} (a_{j_1 i_1}^{(N,N-1)} \dots a_{j_L i_L}^{(N,N-1)}) \quad (\text{B.3})$$

$$= \prod_{r=1}^{N-1} \text{Tr} \left[\prod_{l=1}^L a_{j_l i_l}^{(N,r)} \right]. \quad (\text{B.4})$$

Representation (B.4) shows that an element of the transfer matrix can be written as a product of $N - 1$ traces of products of L fundamental matrices $\{\delta, \epsilon, A, \mathbb{1}\}$. In the following, we shall call $\text{Tr} \left[\prod_{l=1}^L a_{j_l i_l}^{(N,r)} \right]$ the r th trace of $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$. As $T_L^{(N)}$ is expressed in terms of traces of elements of \mathcal{F} , we have to study which elements of \mathcal{F} have zero trace and which have nonzero trace in order to determine which are the nonzero matrix elements of $T_L^{(N)}$. In appendix A we showed, using the quadratic algebra (19)–(21), that the trace of a product of elements of \mathcal{F} with at least one A is nonzero (for $q \neq 0$) if and only if the number of δ is the same as the number of ϵ in the product. In terms of the matrix elements of the transfer matrix $T_L^{(N)}$, this means that each of the $N - 1$ traces of $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ must contain the same number of δ and ϵ as we now show.

The condition of having at least one A in each trace is automatically verified if we choose the configuration \vec{j} such that it contains at least one particle of class N because a factor $a_{N0}^{(N)}$ appears (33). From expressions (27)–(33) for the $a_{j_l i_l}^{(N)}$, we see that a δ appears in the r th trace of $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ for all l such that $j_l = r$ and either $i_l = 0$ or $i_l > r$. This corresponds to sites at which a hole or a particle of class strictly larger than r is replaced by a particle of class r . On the other hand, an ϵ appears in the r th trace of $\langle \vec{j} | T_L^{(N)} | \vec{i} \rangle$ in expression (B.4) for all l such that

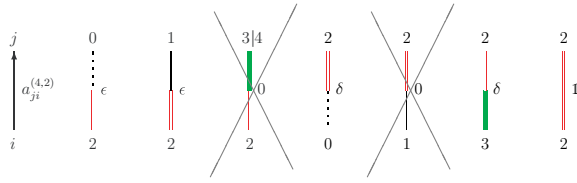


Figure B1. All the situations involving a particle of class $r = 2$ for $N = 4$. A particle of class 0 is represented by a vertical dashed line, a particle of class 1 by a full line, a particle of class 2 by a double line and a particle of class 3 or 4 by a triple line. The crossed-out diagrams correspond to situations which are forbidden by (27).

$i_l = r$ and $0 \leq j_l < r$. This corresponds to sites at which a particle of class r becomes a hole or a particle of class strictly lower than r . For example for $N = 2$, δ correspond to $0 \rightarrow 1$ and ϵ to $1 \rightarrow 0$ (see figure 1). For $N = 3$, a δ appears in the first trace for $0 \rightarrow 1$ and $2 \rightarrow 1$ and an ϵ appears for $1 \rightarrow 0$, and there is a δ in the second trace for $0 \rightarrow 2$ and an ϵ for $2 \rightarrow 0$ and $2 \rightarrow 1$ (see figure 2). In figure B1, we draw all the transitions involving a particle of class 2 when $N = 4$ (those which are forbidden are crossed out), along with the corresponding values for $a_{ji}^{(4,2)}$ (second trace). To summarize, in the r th trace:

- a δ appears when a particle of class $0, r + 1, r + 2, \dots, N - 1$ in \vec{i} is transformed into a particle of class r at the same site in \vec{j} ,
- an ϵ appears when a particle of class of class r in \vec{i} is transformed into a particle of class $0, 1, 2, \dots, r - 1$, at the same site in \vec{j} .

Therefore, the requirement that the number of δ and ϵ is the same in the r th trace of $T_L^{(N)}$ in (B.4) implies that the number of particles of class r between 1 and $N - 1$ is conserved by the action of the transfer matrix $T_L^{(N)}$. But we emphasize that neither the number of holes nor the number of particles of class N is conserved: the number of holes decreases while the number of particles of class N increases. This concludes the proof of the characterization of the nonzero matrix elements of the transfer matrix $T_L^{(N)}$.

Appendix C. Proofs of equations (70) and (71) for $K = 0 < J$

The commutation relations between $a_{JM}^{(N)}$ and $a_{KM'}^{(N)}$ were given in section 5 for the case $0 < J < K$. They were used in section 6 to prove equations (70) and (71) for $0 < J < K$. In this appendix, we will write the commutation relations between $a_{JM}^{(N)}$ and $a_{0M'}^{(N)}$, and use them to prove equations (70) and (71) for $K = 0 < J$.

C.1. Commutation relations

We partition the ensemble of couples (M, M') for which $a_{JM}^{(N)}$ and $a_{0M'}^{(N)}$ are defined and nonzero (that is $M \in \{0\} \cup [J, N - 1]$ and $M' \in [0, N - 1]$) into 12 sectors as

	$M' = 0$	$0 < M' < J$	$M' = J$	$J < M' \leq N - 1$	
$M = 0$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\bar{10}$	(C.1)
$M = J$	$\bar{2}$	$\bar{5}$	$\bar{8}$	$\bar{11}$	
$J < M \leq N - 1$	$\bar{3}$	$\bar{6}$	$\bar{9}$	$\bar{12}$.	

Then, we have the following commutation relations between $a_{JM}^{(N)}$ and $a_{0M'}^{(N)}$:

$$a_{JM}^{(N)}a_{0M'}^{(N)} = qa_{0M'}^{(N)}a_{JM}^{(N)} \quad \text{in sectors } \bar{4}, \bar{5}, \bar{6} \quad (\text{C.2})$$

$$a_{JM}^{(N)}a_{0M'}^{(N)} = a_{0M'}^{(N)}a_{JM}^{(N)} \quad \text{in sectors } \bar{1}, \bar{2}, \bar{3}, \bar{8}, \bar{10}, \bar{11} \text{ and } \bar{12} \quad (\text{C.3})$$

$$a_{JM}^{(N)}a_{0M'}^{(N)} = qa_{0M'}^{(N)}a_{JM}^{(N)} + (1 - q)a_{0M'}^{(N)}a_{JM}^{(N)} \quad \text{in sectors } \bar{7} \text{ and } \bar{9}. \quad (\text{C.4})$$

C.2. Proof of equation (70) ($K = 0 < J$)

Let us define

$$\bar{\mathcal{A}} = X_J^{(N)}X_0^{(N)} - qX_0^{(N)}X_J^{(N)} - \hat{X}_J^{(N)}X_0^{(N)} + X_J^{(N)}\hat{X}_0^{(N)}. \quad (\text{C.5})$$

We want to show that $\bar{\mathcal{A}} = 0$. With the extra term $-(1 - q)X_0^{(N)}$ that comes in the recursive expression for $\hat{X}_0^{(N)}$, the expression for $\bar{\mathcal{A}}$ rewrites

$$\begin{aligned} \bar{\mathcal{A}} = & \sum_{M \in \{0\} \cup [J, N-1]} \sum_{M'=0}^{N-1} [q(a_{JM}a_{0M'}) \otimes (X_M X_{M'}) - q(a_{0M'}a_{JM}) \otimes (X_{M'} X_M) \\ & - (a_{JM}a_{0M'}) \otimes (\hat{X}_M X_{M'}) + (a_{JM}a_{0M'}) \otimes (X_M \hat{X}_{M'})]. \end{aligned}$$

Like in the case $0 < J < K$, we will cut the double sum into four parts and write $\bar{\mathcal{A}} = \bar{\mathcal{A}}_1 + \bar{\mathcal{A}}_2 + \bar{\mathcal{A}}_3 + \bar{\mathcal{A}}_4$, gathering sectors from the partition (C.1). $\bar{\mathcal{A}}_1$ will be made of the sectors $\bar{4}, \bar{5}$ and $\bar{6}$ of (C.1), $\bar{\mathcal{A}}_2$ of the sectors $\bar{1}, \bar{3}, \bar{10}$ and $\bar{12}$, $\bar{\mathcal{A}}_3$ of the sectors $\bar{2}, \bar{7}, \bar{9}$ and $\bar{11}$, and $\bar{\mathcal{A}}_4$ of the sector $\bar{8}$:

$$\bar{\mathcal{A}}_1 : \quad M \in \{0\} \cup [J, N - 1] \quad \text{and} \quad M' \in [1, J - 1] \quad (\text{C.7})$$

$$\bar{\mathcal{A}}_2 : \quad M \in \{0\} \cup [J + 1, N - 1] \quad \text{and} \quad M' \in \{0\} \cup [J + 1, N - 1] \quad (\text{C.8})$$

$$\bar{\mathcal{A}}_3 : \quad \begin{cases} M = J & \text{and} \quad M' \in \{0\} \cup [J + 1, N - 1] \\ M \in \{0\} \cup [J + 1, N - 1] & \text{and} \quad M' = J \end{cases} \quad (\text{C.9})$$

$$\bar{\mathcal{A}}_4 : \quad M = J \quad \text{and} \quad M' = J. \quad (\text{C.10})$$

We will now show that $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3$ and $\bar{\mathcal{A}}_4$ are all equal to zero.

We begin with $\bar{\mathcal{A}}_1$ and use the commutation relation (C.2). We get

$$\bar{\mathcal{A}}_1 = \sum_{M=0}^{N-1} \sum_{M'=1}^{J-1} (a_{JM}a_{0M'}) \otimes (qX_M X_{M'} - X_{M'} X_M - \hat{X}_M X_{M'} + X_M \hat{X}_{M'}). \quad (\text{C.11})$$

But, from (71), $qX_M X_{M'} - X_{M'} X_M - \hat{X}_M X_{M'} + X_M \hat{X}_{M'} = 0$ by induction. Thus, $\bar{\mathcal{A}}_1 = 0$.

For $\bar{\mathcal{A}}_2$, using the commutation relation (C.3), we obtain

$$\bar{\mathcal{A}}_2 = \sum_{M, M' \in \{0\} \cup [J+1, N-1]} (a_{JM}a_{0M'}) \otimes (q[X_M, X_{M'}] - \hat{X}_M X_{M'} + X_M \hat{X}_{M'}). \quad (\text{C.12})$$

From (58), $a_{JM}a_{0M'}$ is symmetric under the exchange of M and M' while $[X_M, X_{M'}]$ is antisymmetric, as well as $-\hat{X}_M X_{M'} + X_M \hat{X}_{M'}$ because of (72) by induction. This gives $\bar{\mathcal{A}}_2 = 0$.

For $\bar{\mathcal{A}}_3$, using the commutation relations (C.3) and (C.4), we have

$$\begin{aligned} \bar{\mathcal{A}}_3 = & \sum_{M' \in \{0\} \cup [J+1, N-1]} (a_{JJ} a_{0M'}) \otimes (q[X_J, X_{M'}] - \hat{X}_J X_{M'} + X_J \hat{X}_{M'}) \\ & + \sum_{M \in \{0\} \cup [J+1, N-1]} [(a_{JM} a_{0J}) \otimes (q X_M X_J - X_J X_M - \hat{X}_M X_J + X_M \hat{X}_J) \\ & + (1 - q)(a_{0M} a_{JJ}) \otimes (X_J X_M)]. \end{aligned} \quad (\text{C.13})$$

By induction, $q X_M X_J - X_J X_M - \hat{X}_M X_J + X_M \hat{X}_J = 0$ in the second sum (71). With the help of the commutation relation (C.3) in sectors $\bar{2}$ and $\bar{11}$ for $a_{0M} a_{JJ}$, we obtain, after renaming the dummy variable M' to M ,

$$\bar{\mathcal{A}}_3 = \sum_{M \in \{0\} \cup [J+1, N-1]} (a_{JJ} a_{0M}) \otimes (X_J X_M - q X_M X_J - \hat{X}_J X_M + X_J \hat{X}_M). \quad (\text{C.14})$$

From (70), $X_J X_M - q X_M X_J - \hat{X}_J X_M + X_J \hat{X}_M = 0$ by induction. $\bar{\mathcal{A}}_3$ is then equal to 0.

Finally, for $\bar{\mathcal{A}}_4$,

$$\bar{\mathcal{A}}_4 = q(a_{JJ} a_{0J} - a_{0J} a_{JJ}) \otimes (X_J X_J) + (a_{JJ} a_{0J}) \otimes (-\hat{X}_J X_J + X_J \hat{X}_J). \quad (\text{C.15})$$

In the first term $a_{JJ} a_{0J} - a_{0J} a_{JJ} = 0$ because of the commutation relation (C.3), and in the last term $-\hat{X}_J X_J + X_J \hat{X}_J = 0$ by induction (69). Thus $\bar{\mathcal{A}}_4 = 0$, which proves (70) in the case $K = 0 < J$.

C.3. Proof of equation (71) ($K = 0 < J$)

Let us define

$$\bar{\mathcal{B}} = X_J^{(N)} X_0^{(N)} - q X_0^{(N)} X_J^{(N)} - X_0^{(N)} \hat{X}_J^{(N)} + \hat{X}_0^{(N)} X_J^{(N)}. \quad (\text{C.16})$$

We want to show that $\bar{\mathcal{B}} = 0$. With the extra term $-(1 - q)X_0^{(N)}$ that comes in the recursive expression for $\hat{X}_0^{(N)}$, the expression for $\bar{\mathcal{B}}$ rewrites

$$\begin{aligned} \bar{\mathcal{B}} = & \sum_{M \in \{0\} \cup [J, N-1]} \sum_{M'=0}^{N-1} [(a_{JM} a_{0M'}) \otimes (X_M X_{M'}) - (a_{0M'} a_{JM}) \otimes (X_{M'} X_M) \\ & - (a_{0M'} a_{JM}) \otimes (X_{M'} \hat{X}_M) + (a_{0M'} a_{JM}) \otimes (\hat{X}_{M'} X_M)]. \end{aligned} \quad (\text{C.17})$$

Again, we cut the double sum into four parts and write $\bar{\mathcal{B}} = \bar{\mathcal{B}}_1 + \bar{\mathcal{B}}_2 + \bar{\mathcal{B}}_3 + \bar{\mathcal{B}}_4$, gathering sectors from the partition (C.1). $\bar{\mathcal{B}}_1$ will be made of the sectors $\bar{4}$, $\bar{5}$ and $\bar{6}$ of (C.1), $\bar{\mathcal{B}}_2$ of the sectors $\bar{1}$, $\bar{3}$, $\bar{10}$ and $\bar{12}$, $\bar{\mathcal{B}}_3$ of the sectors $\bar{2}$, $\bar{7}$, $\bar{9}$ and $\bar{11}$ and $\bar{\mathcal{B}}_4$ of the sector $\bar{8}$. Using the same arguments as in appendix C.2 we can show that $\bar{\mathcal{B}}_1$, $\bar{\mathcal{B}}_2$, $\bar{\mathcal{B}}_3$ and $\bar{\mathcal{B}}_4$ are all equal to zero, which proves (71) in the case $K = 0 < J$.

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